Supplemental Material:
Cooperative emission of a pulse train in an optically thick scattering medium

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Optical thickness measurement

We employ three different methods to measure the optical thickness. First, we compute the theoretical transmission spectrum for various $b_{\bar{v}}(\delta)$ values and use these profiles to fit the experimentally obtained transmission data. This leads to an optical thickness $b_{\bar{v}}(0) = 19$. Second, we perform a shadow imaging experiment on the $^1S_0 \rightarrow {}^1P_1$ broad transition ($\lambda_b = 461$ nm, linewidth $\Gamma_b = 2\pi \times 32$ MHz), where Doppler broadening is negligible. A collimated probe beam with a waist larger than the atomic cloud is sent onto the cloud, and the transmission signal I_t/I_0 is measured using an electron multiplying CCD camera (*Andor iXon Ultra 897*). Typically, the probe frequency is set at a detuning, $\delta_b = 53$ MHz, to reduce the systematic error in the transmission measurement due to large optical thickness. The optical thickness $\mathcal{B}$ is computed from the transmission signal, $\mathcal{B} = -\log(I_t/I_0)$, and is related to $b_0(0)$ of the intercombination line by $b_0(0) = \mathcal{B}\left(1 + 4\delta_b^2/\Gamma_b^2\right)\lambda^2/\lambda_b^2$. In our experiment, we measure a peak value of $b_0(0) = 95(5)$ using this method and a corresponding value of $b_{\bar{v}}(0) = 15(1)$ using $b_{\bar{v}}(0) = b_0(0)g(k\bar{v}/\Gamma)$. Third, we carry out shadow imaging experiment directly on the intercombination line transition. We vary the detuning in a range of 100 kHz around the resonance. The value of $b_{\bar{v}}(0)$ is deduced using

$$E_t(\omega) = E_0(\omega)\mathrm{e}^{i\frac{n(\omega)\omega L}{c}}, \tag{S1}$$

and

$$\alpha(\omega) = -\frac{3\pi\Gamma c^3}{\omega^3}\frac{1}{\sqrt{2\pi}\bar{v}}\int_{-\infty}^{+\infty}\mathrm{d}v\frac{\mathrm{e}^{-v^2/2\bar{v}^2}}{\delta - kv + i\Gamma/2}, \tag{S2}$$

which are Eqs. (1) and (2) in the main text. We have $b_{\bar{v}}(0) = 19(2)$, a value slightly larger than the one obtained by the second method.

Initial decay time $\tau_{\bar{v}}$

We take $t = 0$ as the time when the abrupt change occurs for the incident field E_0. To calculate the initial decay time of the cooperative forward transmitted field, we first note that we can rewrite Eq. (5) in the main text as

$$\tau_{\bar{v}}(\delta) = \left|\frac{1 - |E_t(t=\infty)|^2/|E_t(t=0^+)|^2}{2\,\mathrm{Re}\left\{\frac{\mathrm{d}E_t/\mathrm{d}t(t=0^+)}{E_t(t=0^+)}\right\}}\right|, \tag{S3}$$

where $E_t(t=0^+) = E_0(t=0^+) + E_s(t=0^-)$, and $E_t(t=\infty)$ is the steady state transmitted field after the abrupt change in the incident field. For the case of abrupt extinction, $E_t(t=0^+) = E_s(t=0^-)$ and $E_t(t=\infty) = 0$. For abrupt ignition, $E_t(t=0^+) = E_0$ and $|E_t(t=\infty)| = |E_0|\mathrm{e}^{-b/2}$. Here, $b = b_v(\delta)$ and $\theta = \theta_v(\delta)$, the same as defined in Eq. (3) of the main text:

$$b_{\bar{v}}(\delta) = \frac{2\omega}{c}\,\mathrm{Im}[n(\omega)]L, \quad \theta_{\bar{v}}(\delta) = \frac{\omega}{c}\,\mathrm{Re}[n(\omega)-1]L. \tag{S4}$$

For abrupt phase jump by φ, we have $E_t(t=0^+) = E_0\mathrm{e}^{i\varphi} + E_s(t=0^-)$, ignoring the small propagation time L/c in the medium, and $|E_t(t=\infty)| = |E_0|\mathrm{e}^{-b/2}$. The forward scattered field during the steady state regime, in both cases of abrupt extinction and phase jump, is $E_s(t=0^-) = E_0\mathrm{e}^{-b/2+i\theta} - E_0$.

In the denominator of Eq. (S3), we need to compute the time derivative of the transmitted field at $t = 0^+$. It can be computed by considering the derivative $\mathrm{d}\left[E_s(t)\mathrm{e}^{i\omega t}\right]/\mathrm{d}t$. The forward scattered field in the time domain, $E_s(t)$, is related to the incident field in the frequency domain, $E_0(\omega')$, by the following well-behaved integral:

$$E_s(t) = \int \mathrm{e}^{-i\omega' t}\left[\mathrm{e}^{i\frac{\omega'\rho\alpha(\omega')L}{2c}} - 1\right]E_0(\omega')\mathrm{d}\omega'. \tag{S5}$$

The integration ranges of the integrals in this Supplemental Material, when not specified, are from $-\infty$ to ∞. $E_0(\omega')$ is given for the cases of abrupt ignition, abrupt extinction and abrupt phase jump of φ by:

$$E_0(\omega') = \frac{i\xi E_0}{2\pi}\mathrm{PV}\frac{1}{\omega' - \omega} + \frac{\eta E_0}{2}\delta(\omega' - \omega). \tag{S6}$$

where ω is the frequency of the probe. The Fourier variable corresponding to t is denoted as ω'. ξ and η are -1

and 1 respectively for abrupt extinction of the probe, and $e^{i\varphi}-1$ and $1+e^{i\varphi}$ respectively for abrupt phase jump of the probe field. In the case of abrupt probe ignition, both ξ and η are equal to 1. We substitute Eq. (S6) in Eq. (S5), noting that the integral involving the Dirac delta function goes to zero, to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[E_s(t)\mathrm{e}^{i\omega t}\right]\\&=\frac{\xi E_0}{2\pi}\sum_{p=1}^{\infty}\frac{1}{p!}\int\left(\frac{i\omega'\rho\alpha(\omega')L}{2c}\right)^p\mathrm{e}^{-i(\omega'-\omega)t}\mathrm{d}\omega'.\end{aligned}\tag{S7}$$

We work in the regime where $\delta,\Gamma,k\bar{v}\ll\omega_0$. For $p=1$, the integral can be evaluated to be

$$\begin{aligned}&\frac{\xi E_0}{2\pi}\int\frac{i\omega'\rho\alpha(\omega')L}{2c}\mathrm{e}^{-i(\omega'-\omega)t}\mathrm{d}\omega'\\&=\frac{\xi E_0}{2\pi i}\frac{b_0(0)}{2}\frac{\Gamma}{2}\frac{1}{\sqrt{2\pi}\bar{v}}\iint\mathrm{d}v\,\mathrm{d}\omega'\frac{\mathrm{e}^{-i(\omega'-\omega)t}\mathrm{e}^{-v^2/2\bar{v}^2}}{\omega'-\omega_0-kv+i\Gamma/2}\\&=-\xi E_0\frac{b_0(0)\Gamma}{4}\mathrm{e}^{i\delta t}\mathrm{e}^{-\Gamma t/2}\frac{1}{\sqrt{2\pi}\bar{v}}\int\mathrm{d}v\,\mathrm{e}^{-ikvt}\mathrm{e}^{-v^2/2\bar{v}^2}\\&=-\xi E_0\frac{b_0(0)\Gamma}{4}\mathrm{e}^{i\delta t}\mathrm{e}^{-\Gamma t/2}\mathrm{e}^{-k^2\bar{v}^2t^2/2},\end{aligned}\tag{S8}$$

for $t>0$. In Eq. (S8), $b_0(0)=6\pi\rho c^2L/\omega_0^2$. The $p>1$ terms, in general, are difficult to evaluate for the general time dependence. Nevertheless, at $t=0^+$, they vanish. We take the example of the term $p=2$, where essentially we have to deal with the following triple integral.

$$\iiint\frac{\mathrm{e}^{-i(\omega-\omega')t}\mathrm{e}^{-v^2/(2\bar{v}^2)}\mathrm{e}^{-v'^2/(2\bar{v}^2)}\,\mathrm{d}\omega'\mathrm{d}v\,\mathrm{d}v'}{(\omega'-\omega_0-kv+i\Gamma/2)(\omega'-\omega_0-kv'+i\Gamma/2)}.\tag{S9}$$

We rewrite for $v\neq v'$,

$$\begin{aligned}&\frac{\mathrm{e}^{-i(\omega-\omega')t}}{(\omega'-\omega_0-kv+i\Gamma/2)(\omega'-\omega_0-kv'+i\Gamma/2)}\\&=\frac{1}{k(v-v')}\frac{\mathrm{e}^{-i(\omega-\omega')t}}{\omega'-\omega_0-kv+i\Gamma/2}\\&\quad-\frac{1}{k(v-v')}\frac{\mathrm{e}^{-i(\omega-\omega')t}}{\omega'-\omega_0-kv'+i\Gamma/2}.\end{aligned}\tag{S10}$$

The integration over ω' of the above expression can be carried out easily, which results in 0 at $t=0^+$. When $v=v'$, we have an integral over a multiple pole of order 2, which also goes to 0 at $t=0^+$. Therefore, the term with $p=2$ is zero at $t=0^+$. Similar argument can be extended to all orders $p>1$, showing that all $p>1$ terms vanish at $t=0^+$. Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[E_s(t)\mathrm{e}^{i\omega t}\right](t=0^+)=-\xi E_0\frac{b_0(0)\Gamma}{4}.\tag{S11}$$

We then use the fact that $E_s(t=0^+)=E_t(t=0^+)-E_0(t=0^+)$ to obtain

$$\frac{\mathrm{d}E_t}{\mathrm{d}t}(t=0^+)=-\xi E_0\frac{b_0(0)\Gamma}{4}-i\omega E_t(t=0^+).\tag{S12}$$

Using the above expression, we deduce the initial decay time for the case of abrupt probe extinction,

$$\tau_{\bar{v}}(\delta)=\frac{2}{\Gamma b_0(0)}\frac{1+\exp(-b)-2\exp(-b/2)\cos(\theta)}{1-\exp(-b/2)\cos(\theta)},\tag{S13}$$

which is Eq. (5) of the main text. For the case of abrupt phase change, we find the initial decay time to be

$$\tau_{\bar{v}}(\delta)=\frac{4}{\Gamma b_0(0)}\frac{1-\exp(-b/2)\cos\theta}{2-\exp(-b/2)\cos\theta}.\tag{S14}$$

This is Eq. (6) of the main text. The initial decay time of the flash in the case of abrupt ignition is found to be

$$\tau_{\bar{v}}(\delta)=\frac{2}{\Gamma b_0(0)}\left[1-\mathrm{e}^{-b}\right].\tag{S15}$$

We observe again the appearance of the factor $2/\Gamma b_0(0)$ which arises from the cooperativity among the atomic dipoles.

In the case of an abrupt phase jump, we can further choose in the experiment, for φ to be equal to the phase of $E_s(t=0^-)$ relative to $E_0(t=0^-)$. This choice ensures a constructive interference after the phase jump. The decay time can be simplified to

$$\tau_{\bar{v}}(\delta)=\frac{4}{\Gamma b_0(0)}\frac{|E_s(t=0^+)|}{E_0+|E_s(t=0^+)|}.\tag{S16}$$